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$3d$ CFT and multi M2-brane theory on $R \times S^2$

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Abstract: The radial quantization of $\mathcal{N}=8$ theory in three dimensions is considered i.e. we study the $\mathcal{N}=8$ BLG theory on $R \times S^2$. We present the explicit from of the Lagrangian and the corresponding supersymmetry transformations and supersymmetry algebra. We study spectrum of this theory and some of its BPS configurations.

Keywords: AdS-CFT Correspondence, M-Theory

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1 Introduction

The world volume theory of multiple M2-branes has been an open question over the last two-three decades ago. The low energy limit of multiple M2-branes theory is expected to be an interacting 2+1 dimensional superconformal field theory with eight transverse scalar fields as its bosonic content [1]. Moreover the multiple M2-branes theory should be maximally supersummetric, which in three dimensions means that it is "$\mathcal{N} = 8$ supersymmetric theory" and therefore superconformal symmetry group is $OSp(8|4)$. From AdS/CFT point of view these theories are dual to $AdS_4 \times S^7$ solution of M-theory. The bosonic part of superalgebra is $\mathrm{SO}(8) \times \mathrm{SO}(3,2)$ as the global symmetry of both theories.

Very interesting theory with appropriate symmetries of $3d$ $\mathcal{N} = 8$ was proposed by Bagger and Lambert [2–4] and also Gustavsson [5]. Therefore this model has potential to describe world volume of multiple M2-branes. In this construction the field content is a collection of eight scalars, fermions and non-propagating gauge fields which are transforming under 3-algebra and a 4-index structure constant. 3-algebra and structure constants can be considered as a generalization of a Lie algebra with triple bracket and 3-index structure constant. The structure constants satisfy a fundamental identity replacing the Jacobi identity of a Lie algebra. There should also be a symmetric invertible metric h^{ab} that can be used to raise and lower indices. Different aspects of this theory are studied in the literature [22]. Matrix realization of this theory is also presented in [6–9].

According to AdS/CFT and holographic principle this model lives on the boundary of $AdS_4 \times S^7$. It is well known that this boundary is $R \times S^2$. In this paper we study

$\mathcal{N}=8$ BLG model on the $R\times S^2$ background. we construct suitable supersymmetry transformations and Lagrangian and then investigate BPS configurations.

This paper is organized as follows. In section 2 we review BLG model and then in section 3 theory on $R\times S^2$ is considered. In section 4 we study BPS configurations. Section 5 is devoted to discussions. The explicit representations for gamma matrices have been presented in appendix A.

2 Review of BLG theory

To begin, we briefly review Bagger-Lambert construction as a three dimensional superconformal field theory with $OSp(8|4)$ superalgebra. $spin(8)$ R-symmetry and $spin(4)\equiv spin(3,2)$ conformal symmetry are the bosonic part of superalgebra. Bosonic fields are X^I_a and non-propagating $A^{ab}_\mu (\mu=0,1,2$ is world volume index) and fermionic fields are Ψ^a. The index I labels components of the fundamental $\mathbf{8}_v$ representation of $spin(8)$ as scalar fields corresponding to the eight directions transverse to M2-branes and a indices take the values $1,\ldots,dim_{\mathcal{A}}$ with $dim_{\mathcal{A}}$ being the dimension of 3-algebra $\mathcal{A}$ which is yet to be specified. Representation of the fermionic fields are $\mathbf{8}_s$ and they have two different indices related to $spin(8)\times spin(3,2)$ which are suppressed here.

In order to write Lagrangian 4-index structure constants f^{abcd} is defined associated with a formal, totally antisymmetric three bracket over 3-algebra generators

$$[T^a,T^b,T^c]=f^{abc}{}_dT^d, \tag{2.1}$$

and inner product is defined by a generalization of the trace over the 3-algebra indices

$$h^{ab}=Tr(T^aT^b). \tag{2.2}$$

The 4-index structure constants satisfy the "fundamental identity"

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d. \tag{2.3}$$

The above bracket and trace satisfy the

$$Tr\left([T^a,T^b,T^c]T^d\right) = -Tr\left([T^d,T^a,T^b]T^c\right), \tag{2.4}$$

implying

$$f^{abcd}=f^{[abcd]}, \tag{2.5}$$

where $f^{abcd}=f^{abc}_e h^{ed}$. The BLG Lagrangian

$$\begin{aligned}\mathcal{L}=&-\frac{1}{2}(D_\mu X^{aI})(D^\mu X^I_a)+\frac{i}{2}\bar\Psi^a\gamma^\mu D_\mu\Psi_a+\frac{i}{4}\bar\Psi_b\Gamma^{IJ}X^I_cX^J_d\Psi_a f^{abcd}\\ &-V+\frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd}+\frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right),\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}V&=\frac{1}{12}f^{abcd}f^{efg}{}_dX^I_aX^J_bX^K_cX^I_eX^J_fX^K_g\\ (D_\mu X)_a&=\partial_\mu X_a - f^{cdb}{}_aA_{\mu\ cd}X_b\equiv\partial_\mu X_a-\tilde A_\mu{}^b{}_aX_b,\end{aligned} \tag{2.7}$$

is invariant under the gauge transformations

$$\begin{aligned}
\delta X_a &= \Lambda_{cd} f^{cdb}{}_a X_b \equiv \tilde{\Lambda}^b{}_a X_b \\
\delta \Psi_a &= \Lambda_{cd} f^{cdb}{}_a \Psi_b \\
\delta(f^{cdb}{}_a A_{\mu cd}) \equiv \delta \tilde{A}_\mu{}^b{}_a &= f^{cdb}{}_a D_\mu \Lambda_{cd},
\end{aligned} \tag{2.8}$$

and the supersymmetry variations

$$\delta X^I_a = i\bar{\epsilon}\Gamma^I \Psi_a \tag{2.9a}$$

$$\delta \Psi_a = D_\mu X^I_a \gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK} \epsilon \tag{2.9b}$$

$$\delta \tilde{A}_\mu{}^b{}_a = i\bar{\epsilon}\gamma_\mu \Gamma^I X^I_c \Psi_d f^{cdb}{}_a. \tag{2.9c}$$

In the above Ψ and ϵ should have different $3d$ chirality i.e. $\gamma^{012}\Psi = -\Psi$ and $\gamma^{012}\epsilon = \epsilon$. It was shown in [3, 4] that the above supersymmetry transformations are closed up to a gauge transformation

$$[\delta_1, \delta_2] X^I_a = v^\mu \partial_\mu X^I_a + (\tilde{\Lambda}^b{}_a - v^\nu \tilde{A}_\nu{}^b{}_a) X^I_b \tag{2.10a}$$

$$[\delta_1, \delta_2] \Psi_a = v^\mu \partial_\mu \Psi_a + (\tilde{\Lambda}^b{}_a - v^\nu \tilde{A}_\nu{}^b{}_a) \Psi_b \tag{2.10b}$$

$$[\delta_1, \delta_2] \tilde{A}_\mu{}^b{}_a = v^\nu \partial_\nu \tilde{A}_\mu{}^b{}_a + D_\mu(\tilde{\Lambda}^b{}_a - v^\nu \tilde{A}_\nu{}^b{}_a), \tag{2.10c}$$

where

$$v^\mu = -2i\bar{\epsilon}_2 \gamma^\mu \epsilon_1, \qquad \tilde{\Lambda}^b{}_a = -i\bar{\epsilon}_2 \Gamma^{JK} \epsilon_1 X^J_c X^K_d f^{cdb}{}_a. \tag{2.11}$$

It is important to notice that the fundamental identity is essential to ensure the gauge invariance of the action as well as the closure of supersymmetry transformations (2.10c). Note also that the supersymmetry transformations (2.10) are written on-shell with the following equations of motion

$$\begin{aligned}
\gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma^{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a &= 0 \\
D^2 X^I_a - \frac{i}{2}\bar{\Psi}_c \Gamma^{IJ} X^J_d \Psi_b f^{cdb}{}_a - \frac{\partial V}{\partial X^{Ia}} &= 0 \\
\tilde{F}_{\mu\nu}{}^b{}_a + \varepsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d + \frac{i}{2}\bar{\Psi}_c \gamma^\lambda \Psi_d) f^{cdb}{}_a &= 0,
\end{aligned} \tag{2.12}$$

that

$$\tilde{F}_{\mu\nu}{}^b{}_a = \partial_\nu \tilde{A}_\mu{}^b{}_a - \partial_\mu \tilde{A}_\nu{}^b{}_a - \tilde{A}_\mu{}^b{}_c \tilde{A}_\nu{}^c{}_a + \tilde{A}_\nu{}^b{}_c \tilde{A}_\mu{}^c{}_a. \tag{2.13}$$

3 BLG construction on $R \times S^2$

To construct the BLG theory on $R \times S^2$, we follow the same procedure as in [3, 4]. We propose appropriate supersymmetry transformations and check their closure. As we will see this fixes all the freedom in the choice of the coefficients in the supersymmetry variations as well as the equations of motion. For the "appropriate " supersymmetry variations we need to work with spinors on $R \times S^2$ which in its own turn is constructed using the AdS_4 fermions. As a result we will show that supersymmetry closure again demands fundamental identity and as expected the equation of motion for the X^I_a acquires a mass term.

3.1 Killing spinor on $R \times S^2$

Killing spinor equation on $R \times S^2$ is our aim in this subsection. The relation between Killing spinor on AdS_5 and $R \times S^3$ has been considered in [11]. Here we follow the same way to find Killing spinor on $R \times S^2$. In the global coordinate the metric of AdS_4 with radius a takes the form

$$ds^2 = a^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2), \tag{3.1}$$

and Killing spinors defined by

$$\tilde{\nabla}_{\bar{\mu}}\epsilon = \left(\nabla_{\bar{\mu}} - \frac{1}{2R}\gamma_{\bar{\mu}}\right)\epsilon = 0. \tag{3.2}$$

$\bar{\mu}(=t, \rho, i)$ labels components of AdS_4 metric where i denotes the direction of S^2. Supersymmetry parameters are chiral $4d$ fermions i.e. $\gamma^{\hat{0}\hat{1}\hat{2}\hat{3}}\epsilon = -\epsilon$ which have *four complex.*[1] Covariant derivative is defined by $\nabla_{\bar{\mu}} = \partial_{\bar{\mu}} - \frac{1}{4R}\Omega_{\bar{\mu}}^{\hat{a}\hat{b}}\gamma_{\hat{a}\hat{b}}$ that R is radius of 2-sphere and $\Omega^{\hat{a}\hat{b}}$ is the connection 1-form defined by $d\omega^{\hat{a}} + \Omega^{\hat{a}}{}_{\hat{b}} \wedge \omega^{\hat{b}} = 0$ and $\omega^{\hat{a}}$ is the vierbein defined in the usual manner

$$g_{\bar{\mu}\bar{\nu}} = \eta_{\hat{a}\hat{b}}\omega_{\bar{\mu}}^{\hat{a}}\omega_{\bar{\nu}}^{\hat{b}}, \quad \{\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}\} = 2g^{\bar{\mu}\bar{\nu}}, \quad \{\gamma^{\hat{a}}, \gamma^{\hat{b}}\} = 2\eta^{\hat{a}\hat{b}}, \tag{3.3}$$

and $\tilde{\nabla}$ is written as

$$\begin{aligned}
\tilde{\nabla}_t &= \partial_t + \frac{1}{2R}\sinh\rho\gamma_t\gamma_\rho - \frac{1}{2R}\cosh\rho\gamma_t = e^{-\frac{1}{2R}\rho\gamma}\left(\partial_t - \frac{1}{2R}\gamma_t\right)e^{\frac{1}{2R}\rho\gamma} \\
\tilde{\nabla}_i &= \nabla_i + \frac{1}{2R}\cosh\rho\gamma_i\gamma_\rho - \frac{1}{2R}\sinh\rho\gamma_i = e^{-\frac{1}{2R}\rho\gamma}\left(\nabla_i - \frac{1}{2R}\gamma_i\gamma\right)e^{\frac{1}{2R}\rho\gamma} \\
\tilde{\nabla}_\rho &= \partial_\rho - \frac{1}{2R}\gamma_\rho = \partial_\rho + \frac{1}{2R}\gamma.
\end{aligned} \tag{3.4}$$

We have identified $\gamma^\rho = -\gamma^{\hat{0}\hat{1}\hat{2}} \equiv -\gamma$, γ is the three dimensional chirality and $\gamma^2 = 1$. By above identification, three gamma matrices are independent describing gamma matrices on $R \times S^2$. Note that in this setup SO(8) symmetry of the original BLG theory doesn't change. Therefore, the Killing spinor on AdS_4 and Killing spinor on $R \times S^2$ are related by

$$\epsilon_{AdS_4} = e^{-\frac{1}{2R}\rho\gamma}\epsilon_{R\times S^2}, \tag{3.5}$$

where $\epsilon_{R\times S^2}$ satisfies

$$\nabla_\mu\epsilon = \frac{1}{2R}\omega_\mu\epsilon, \tag{3.6}$$

with

$$\begin{aligned}
&\omega_\mu = (\gamma_t, \gamma_i\gamma)\ ,\ i = 1, 2 \\
&\gamma^\nu\nabla_\nu(\gamma^\mu\nabla_\mu\epsilon) = -\frac{1}{4}d(d-2)\epsilon, \quad d = 3
\end{aligned} \tag{3.7}$$

where γ_i are matrices on the S^2 and d is space-time dimension.

[1] Note that after using $\gamma^{\hat{0}\hat{1}\hat{2}\hat{3}}\epsilon = -\epsilon$, supersymmetry parameters have *two complex* fermionic degrees of freedom. By applying $\gamma^{\hat{0}\hat{1}\hat{2}}\epsilon = \epsilon$ (see after (3.11)), the degrees of freedom are *two real*.

3.2 BLG theory on $R \times S^2$

Inspired by the BLG and similar analysis for the $\mathcal{N} = 4$ on $R \times S^3$ [11], we propose the following deformed supersymmetry transformations for the $\mathcal{N} = 8$ theory on $R \times S^2$

$$\delta X^I_a = i\bar{\epsilon}\Gamma^I\Psi_a \tag{3.8a}$$

$$\delta\Psi_a = D_\mu X^I_a\gamma^\mu\Gamma^I\epsilon - \frac{1}{6}X^I_b X^J_c X^K_d f^{bcd}{}_a\Gamma^{IJK}\epsilon + m\,\Gamma^I X^I_a\gamma^\mu\nabla_\mu\epsilon \tag{3.8b}$$

$$\delta\tilde{A}_\mu{}^b{}_a = i\bar{\epsilon}\gamma_\mu\Gamma^I X^I_c\Psi_d f^{cdb}{}_a, \tag{3.8c}$$

where now D_μ is covariant derivative on the $R \times S^2$ including gauge field

$$(D_\mu X)_a = \nabla_\mu X_a - \tilde{A}_\mu{}^b{}_a X_b, \tag{3.9}$$

and m is the dimensionless parameter to be fixed later. Instead of the $3d$ Majorana fermions used in the original BLG analysis the fermionic fields Ψ should be appropriately chosen for the $R \times S^2$ case. We choose Ψ to be a chiral fermion on the S^2 and hence Ψ is a one component complex fermion while also in $\mathbf{8}_s$ of SO(8). For the supersymmetry transformation parameter ϵ is similarly taken to be a chiral Killing spinor on $R \times S^2$.

Closure of the scalar field X^I leads to

$$[\delta_1, \delta_2]X^I_a = v^\mu\partial_\mu X^I_a + (\tilde{\Lambda}^b{}_a - v^\nu\tilde{A}_\nu{}^b{}_a)X^I_b + i\Lambda^{IJ}X^J_a, \tag{3.10}$$

where

$$\Lambda^{IJ} = m\left(\bar{\epsilon}_2\Gamma^{IJ}\gamma^\mu\nabla_\mu\epsilon_1 - \bar{\epsilon}_1\Gamma^{IJ}\gamma^\mu\nabla_\mu\epsilon_2\right)\ . \tag{3.11}$$

In the above $\gamma^{\hat{0}\hat{1}\hat{2}}\Psi = -\Psi$ and $\gamma^{\hat{0}\hat{1}\hat{2}}\epsilon = \epsilon$. The Γ^{IJ} term shows the SO(8) R-symmetry rotation.[2] Closure of supersymmetry over the fermionic fields leads to

$$[\delta_1, \delta_2]\Psi_a = v^\mu\nabla_\mu\Psi_a + (\tilde{\Lambda}^b{}_a - v^\nu\tilde{A}_\nu{}^b{}_a)\Psi_b + \frac{i}{4}\Lambda^{IJ}\Gamma^{IJ}\Psi_a\,, \tag{3.12}$$

provided that the fermionic equations of motion are

$$\gamma^\mu D_\mu\Psi_a + \frac{1}{2}\Gamma^{IJ}X^I_c X^J_d\Psi_b f^{cdb}{}_a = 0\,, \tag{3.13}$$

and that $m = -\frac{1}{3}$. In other words, the supersymmetry closure condition fixes the only free parameter in our model.

As the last closure condition we examine $[\delta_1, \delta_2]\tilde{A}_\mu{}^b{}_a$. Upon employing the fundamental identity,

$$[\delta_1, \delta_2]\tilde{A}_\mu{}^b{}_a = v^\nu\nabla_\nu\tilde{A}_\mu{}^b{}_a + D_\mu(\tilde{\Lambda}^b{}_a - v^\nu\tilde{A}_\nu{}^b{}_a), \tag{3.14}$$

provided that A_μ is satisfying the following equation of motion

$$\tilde{F}_{\mu\nu}{}^b{}_a + \varepsilon_{\mu\nu\lambda}\left(X^J_c D^\lambda X^J_d + \frac{i}{2}\bar{\Psi}_c\gamma^\lambda\Psi_d\right)f^{cdb}{}_a = 0. \tag{3.15}$$

[2] Using antisymmetric property of γ^0,γ^1 and $\bar{\epsilon}_2\gamma^2\epsilon_1 = 0$, one can explicitly show that $\nabla_\mu\Lambda^{IJ} = 0$. It means that the R-symmetry is rigid. Moreover the explicit superalgebra is written in (3.35) and (3.36) in terms of the fields and their momenta.

The above closure conditions establish the supersymmetric invariance of the BLG model on $R\times S^2$ with the above modified supersymmetry transformations. Note that in this case the supersymmetry algebra besides the "translations on $R\times S^2$" (the $\gamma^\mu\nabla_\mu$ term) also involves an SO(8) R-symmetry rotation.

To find bosonic equation of motion, we take the supervariation of the fermion equation of motion. This gives

$$D^2X^I_a-\frac{i}{2}\bar\Psi_c\Gamma^{IJ}X^J_d\Psi_b f^{cdb}{}_a-\frac{1}{4R^2}X^I_a-\frac{\partial V}{\partial X^{Ia}}=0. \tag{3.16}$$

Finally we present an action for this system. The equations of motion can be obtained from the action

$$\begin{aligned}S=\int dtd\Omega_2\sqrt{-g}\bigg(&-\frac{1}{2}(D_\mu X^{aI})(D^\mu X^I_a)+\frac{i}{2}\bar\Psi^a\gamma^\mu D_\mu\Psi_a+\frac{i}{4}\bar\Psi_b\Gamma^{IJ}X^I_cX^J_d\Psi_a f^{abcd}\\&-\frac{1}{8R^2}(X^I_a)^2-V+\frac{1}{2}\varepsilon^{\mu\nu\lambda}(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd}+\frac{2}{3}f^{cda}{}_gf^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef})\bigg),\end{aligned} \tag{3.17}$$

that $d\Omega_2=R^2\sin\theta d\theta d\phi$. It is not hard to check that the action is gauge invariant and supersymmetric under the transformations (3.8).

In original BLG theory, since h^{ab} is positive definite, it was proved in [16] that the theory has unique solution which is

$$f^{abcd}=\epsilon^{abcd}, \tag{3.18}$$

and then the theory has been written as an ordinary gauge theory with gauge group as $\mathrm{SU}(2)\times\mathrm{SU}(2)$ with bifundamental matter [17]. It is evident that in our case the theory has the same structure compared to original theory and therefore it can be simply written as an ordinary gauge theory with the same gauge group. Moreover one expects that the $3d$ $\mathcal{N}=8$ theory is invariant under the $3d$ parity transformations $x_0,x_1\to x_0,x_1$ and $x_2\to -x_2$ in flat space. It was shown [3] that the parity invariance of the twisted Chern-Simon term implies that under parity

$$A_0,A_1\to A_0,A_1,\;\; A_2\to -A_2,\;\; f\to -f. \tag{3.19}$$

Parity invariance of the kinetic terms as well as the interaction terms imply that under parity scalar fields are invariant and for $3d$ fermions

$$\Psi^a\to\gamma^2\Psi^a. \tag{3.20}$$

By exchanging $(x_0,x_1,x_2)\to(t,\phi,\theta)$, parity transformations for the theory on $R\times S^2$ are $t,\phi,\theta\to t,\pi-\phi,\theta$. For the gauge and fermionic fields we have

$$\begin{aligned}&A_t,A_\theta\to A_t,A_\theta,\;\; A_\phi\to -A_\phi,\;\; f\to -f\\&\Psi^a\to\gamma^\phi\Psi^a\\&X^I_a\to X^I_a.\end{aligned} \tag{3.21}$$

The original BLG theory enjoys superconformal symmetry. It was shown that superconformal symmetry can be found by replacing ϵ by $\gamma.x\eta$ and adding an appropriate term i.e. $-X^I\Gamma^I\eta$ to $\delta\Psi_a$ [10]. Therefore from (2.9b) supersymmetry transformation is

$$\delta_\xi\Psi_a \equiv \delta_{susy}\Psi_a = D_\mu X_a^I\gamma^\mu\Gamma^I\xi - \frac{1}{6}X_b^I X_c^J X_d^K f^{bcd}{}_a\Gamma^{IJK}\xi, \tag{3.22}$$

and superconformal transformation is

$$\delta_\eta\Psi_a \equiv \delta_{su.conf.}\Psi_a = D_\mu X_a^I\gamma^\mu\Gamma^I\gamma.x\eta - \frac{1}{6}X_b^I X_c^J X_d^K f^{bcd}{}_a\Gamma^{IJK}\gamma.x\eta - \ \Gamma^I X_a^I\eta, \tag{3.23}$$

where ξ and η are constant spinors. It is easy to write supersymmetry transformations on $R\times S^2$(3.8b) in terms of (3.22) and (3.23), as a combination of $3d$ superPoincare and $3d$ superconformal transformations

$$\delta_\epsilon\Psi_a = \delta_\xi\Psi_a + \delta_\eta\Psi_a. \tag{3.24}$$

This leads

$$\epsilon = \xi + \gamma.x\eta. \tag{3.25}$$

In the original BLG theory ξ and η are $3d$ Majorana fermion in $\mathbf{8}_c$ of SO(8) and then they have $16+16$ degrees of freedom. Supersymmetry transformations on $R\times S^2$ are generated by 16 independent ϵ's. The other combination should be considered as a "*superconformal symmetry*" on $R\times S^2$.

In order to understand the theory we would like to study complete spectrum about $X^I=0$ vacuum. To do so, we expand the theory about the vacua to second order in small fluctuations. Then equations of motion for X_a^I are

$$(\partial_t^2 - \frac{1}{R^2}\nabla_{S^2}^2 + \frac{1}{4R^2})X_a^I = 0, \tag{3.26}$$

where $\nabla_{S^2}^2$ is written on the sphere with radius one. By expanding X_a^I in terms of spherical harmonics on the 2-sphere we have

$$X_a^I = \sum_l x_{a,lm}^I e^{iw_l t}Y_{lm}(\theta,\phi), \tag{3.27}$$

and hence these modes would have mass squared equal to

$$R^2w_l^2 = \left(l+\frac{1}{2}\right)^2, \quad l=0,1,\ldots \tag{3.28}$$

For fermionic fields by using (3.8a) we have

$$\left(\partial_t^2 - \frac{1}{R^2}\nabla_{S^2}^2 + \frac{1}{4R^2}\right)\delta X_a^I = 0, \tag{3.29}$$

which leads[3]

$$\left(\partial_t^2 - \frac{1}{R^2}\nabla_{S^2}^2 + \frac{1}{4R^2}\right)\Psi_a = 0. \tag{3.30}$$

[3] Note that $\nabla^2\epsilon = -\frac{\epsilon}{4R^2}$.

Making the expansion [20]

$$\Psi_a = \sum_j \psi_a^{jm} e^{i\omega_l t} (\sin\theta)^{|m|} Y_{jm}(\theta,\phi), \tag{3.31}$$

where quantum number j is total angular momentum($j = l \pm \frac{1}{2}$) of fermions. Hence mass squared is

$$\begin{aligned} j = l + \frac{1}{2} \quad &: \; R^2\omega_l^2 = (l+1)^2, \quad l = 0, 1, \ldots \\ j = l - \frac{1}{2} \quad &: \; R^2\omega_l^2 = l^2, \qquad\quad l = 1, 2, \ldots \end{aligned} \tag{3.32}$$

As a result of supersymmetry the sum of boson masses and the sum of fermion masses are both $16(l + \frac{1}{2})^2$.

Recently, in [18] an infinite class of brane configurations was given whose low energy effective Lagrangian is a Chern-Simon theory with SO(6) R-symmetry and $\mathcal{N} = 6$ supersymmetry. These theories are related to N M2-branes in R^8/Z_k including $k = 1$. After that by relaxing the condition on three-bracket so that it is no longer real and antisymmetric in all three indices i.e.

$$f^{ab\bar{c}\bar{d}} = -f^{ba\bar{c}\bar{d}}, \quad f^{ab\bar{c}\bar{d}} = f^{*\bar{c}\bar{d}ab}. \tag{3.33}$$

$\mathcal{N} = 6$ theories based on 3-algebra have been obtained [9, 19]. However the new three-bracket is still required to satisfy the fundamental identity. The supersymmetry transformations are [19]

$$\begin{aligned} \delta Z_d^A &= i\bar{\epsilon}^{AB}\psi_{Bd} \\ \delta\psi_{Bd} &= \gamma^\mu D_\mu Z_d^A \epsilon_{AB} + f^{ab\bar{c}}{}_d Z_a^C Z_b^A \bar{Z}_{C\bar{c}}\epsilon_{AB} + f^{ab\bar{c}}{}_d Z_a^C Z_b^D \bar{Z}_{B\bar{c}}\epsilon_{CD} \\ \delta\tilde{A}_\mu{}^c{}_d &= -i\bar{\epsilon}_{AB}\gamma_\mu Z_a^A \psi_b^B f^{ca\bar{b}}{}_d + i\bar{\epsilon}^{AB}\gamma_\mu \bar{Z}_{A\bar{b}}\psi_{Ba} f^{ca\bar{b}}{}_d, \end{aligned} \tag{3.34}$$

where ϵ_{AB} is in the $\mathbf{6}$ of SU(4) and a raised A index indicates that the field is in the $\mathbf{4}$ of SU(4); a lowered index transforms in the $\bar{\mathbf{4}}$. One can write above theory on $R \times S^2$ by adding an appropriate mass term, i.e. $-\frac{1}{3}Z_d^A\gamma^\mu\nabla_\mu\epsilon_{AB}$, in variation of fermionic fields. Since the antisymmetry condition was not used in our earlier supersymmetry closure analysis the above supersymmetry transformations plus mass term will still remain closed. In particular, the closure of the scalar fields will exactly work in the same way as in the $\mathcal{N} = 8$ theory. For the closure of gauge fields, equation of motion, (3.33) and fundamental identity are enough. The closure of fermionic fields just requires fermionic equation of motion. The equation of motion for scalars Z_a^A, as before is found by taking the supervariation of the fermion equation of motion if we apply (3.33). Therefore, one can reproduce the $\mathcal{N} = 6$ supersymmetric theories on $R \times S^2$. (Since the computations are essentially the same as the $\mathcal{N} = 8$ we do not repeat the equations.)

Finally, superalgebra may be written by using (3.10) and (3.12). As we explained before fermionic fields have two different indices relating to SO(3) $\times$ U(1) and SO(8) which is the bosonic part of $OSp(8|2) \times$ U(1) superalgebra. Let's label them with $\dot{\alpha} = 1, 2$ and $\dot{A} = 1, \ldots, 8$ respectively. Then the superalgebra is

$$\{Q_{\dot{\alpha}}^{\dot{A}}, Q_{\dot{\beta}}^{\dot{B}}\} = -2\delta^{\dot{A}\dot{B}}(\gamma^\mu\gamma^0)_{\dot{\alpha}\dot{\beta}}P_\mu + \frac{1}{2}\delta_{\dot{\alpha}\dot{\beta}}(\Gamma^{IJ})^{\dot{A}\dot{B}}J^{IJ}, \tag{3.35}$$

where

$$
\begin{aligned}
J^{IJ} &= \int d\Omega_2\sqrt{-g}\Big(X^I P^J - X^J P^I + \frac{1}{2}\psi^\dagger(i\Gamma^{IJ})\psi\Big)\\
Q &= \int d\Omega_2\sqrt{-g}\Big(D_\mu X^I_a\Gamma^I\gamma^\mu\gamma^0\Psi^a - \frac{1}{6}X^I_b X^J_c X^K_d f^{bcd}{}_a\Gamma^{IJK}\gamma^0\Psi_d + \frac{1}{4R}\,\Gamma^I X^I_a\gamma^0\Psi^a\Big).
\end{aligned}
\tag{3.36}
$$

We have fixed that J^{IJ}'s are SO(8) generators. The superalgebera for original BLG theory has been discussed in [21].

4 BPS solution

By definition a BPS configuration is a state which is invariant under some specific supersymmetry transformations. For the configurations in which spinor fields are turned off the non-vanishing supersymmetry variations are only $\delta_\epsilon\Psi_a$ and hence BPS equations read as

$$\delta_\epsilon\Psi_a = 0, \tag{4.1}$$

for arbitrary ϵ. From the above equation and (3.8b) it is clear that the $X^I = 0$ vacuum is a full BPS configuration(with 32 supercharges). Another class of BPS solutions are small fluctuations about vacuum. In this case the equation (4.1) reads as

$$(\gamma^\mu\nabla_\mu X^I_a\Gamma^I - \frac{1}{2R}X^I_a\Gamma^I)\epsilon = 0, \tag{4.2}$$

where gauge and fermionic fields are turned off. Replacing from (3.27) we have

$$\left(\gamma^t(i\omega_l) + \gamma^i\partial_i - \frac{1}{2R}\right)X^I_a\Gamma^I\epsilon = 0, \quad i = \theta, \phi \tag{4.3}$$

which evidently is right just for $l = 0$ bosonic fluctuations and then they are 1/4 BPS configurations. In this case we have a short multiplet including eight bosonic and four fermionic degrees of freedom. Other possibilities of l are non-BPS solutions with equal number of bosonic and fermionic degrees of freedom. Hence $(l, l+\frac{1}{2}, l+1),\ l > 0$ assemble to a long multiplet. In what follows we discuss other classes of 1/4 BPS configurations.

4.1 1/4 BPS configuration

Let us start with the case in which $X^{5,6,7,8}$'s are turned off and then BPS equation (4.1) takes the form

$$\left(\gamma^\mu D_\mu X^{\hat{i}}\Gamma^{\hat{i}} - \frac{1}{6}[X^{\hat{i}}, X^{\hat{j}}, X^{\hat{k}}]\Gamma^{\hat{i}\hat{j}\hat{k}} - \frac{1}{2R}X^{\hat{i}}\Gamma^{\hat{i}}\right)\epsilon = 0,\ \ \hat{i} = 1,2,3,4. \tag{4.4}$$

In order to solve above equation we introduce

$$X^{\hat{i}} = \alpha\Gamma^{\hat{i}}. \tag{4.5}$$

$\Gamma^{\hat{i}}$'s are in $n \times n$ representation of Spin(4) and obey

$$[\Gamma^{\hat{i}}, \Gamma^{\hat{j}}, \Gamma^{\hat{k}}] = 12\epsilon^{\hat{i}\hat{j}\hat{k}\hat{l}}\Gamma^{\hat{l}}, \tag{4.6}$$

and α is a dimensional constant. Therefore, the first term in (4.4) vanishes and it leads to

$$\left(2.3!\alpha^2\Gamma^5 - \frac{1}{2R}\mathbb{1}\right)X^{\hat{i}}\Gamma^{\hat{i}}\epsilon = 0, \tag{4.7}$$

which has a solution if $\alpha^2 = \frac{1}{24R}$ (Γ^5 is the SO(4) chirality matrix). These solutions are exactly fuzzy three sphere with SO(4) symmetry explained in the literature e.g. [13]. One expects that the theory which lives on the two membranes can be described by BLG theory. It means that in our solution membranes blow up to a fuzzy three sphere in transverse directions. (4.7) shows that ϵ has eight real fermionic degrees of freedom and our solutions are 1/4 BPS. We reproduce trivial case $X^I = 0$ when R goes to infinity.

The other case happens when α is not a constant and can vary on the 2-sphere in the θ direction. We then have

$$\gamma^\theta\partial_\theta X^{\hat{i}}\Gamma^{\hat{i}} - \frac{1}{6}[X^{\hat{i}}, X^{\hat{j}}, X^{\hat{k}}]\Gamma^{\hat{i}\hat{j}\hat{k}} - \frac{1}{2R}X^{\hat{i}}\Gamma^{\hat{i}} = 0. \tag{4.8}$$

It is straightforward to check that the above equation is solved with

$$X^{\hat{i}} = \alpha(\theta)\Gamma^{\hat{i}}, \tag{4.9}$$

provided that

$$\alpha(\theta) = \frac{1}{\sqrt{24s_1R(1 - e^{s_2(\theta-\theta_0)})}}, \tag{4.10}$$

that we have used

$$\begin{aligned}\Gamma^5\epsilon &= s_1\epsilon \\ \gamma^\theta\epsilon &= s_2\epsilon,\end{aligned} \tag{4.11}$$

where s_1 and s_2 can independently be $+1$ or -1. Two different cases exist here which are $e^{s_2(\theta-\theta_0)} > 1, s_1 = -1$ and $e^{s_2(\theta-\theta_0)} < 1, s_1 = +1$. Regarding to the sign of s_2 in each case there are eight independent ϵ's and therefore these configurations are 1/4 BPS. These solutions correspond to M2-brane along $0\theta\phi$ ending on M5-brane along 01234ϕ which means that M5-brane wraps in ϕ direction and as a result there is a U$(1)_\phi$ symmetry. Unlike the previous 1/4 BPS configurations these family of solutions change to Basu-Harvey configurations [14] in specific limit as an open membrane ending on M5-brane(see also [15]). The "Basu-Harvey limit" is then a limit where R is taken to infinity, keeping x finite, i.e.

$$R \to \infty, \quad \theta = x/R, x\ finite, \tag{4.12}$$

and (4.10) becomes

$$\alpha(x) = \frac{1}{\sqrt{-24s_1s_2(x - x_0)}}. \tag{4.13}$$

If $x > x_0$ then we should take $s_1s_2 = -1$ indicating $s_1 = +1, s_2 = -1$ or $s_1 = -1, s_2 = +1$. Each of them preserves four independent ϵ's and we have 1/4 BPS Basu-Harvey configurations. (For the other case, $x < x_0$, there are again eight ϵ's.)

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5 Conclusion

In this work we have generalized the $3d$, $\mathcal{N}=8$ BLG theory on flat space to $R\times S^2$. As we discussed an additional term adds to supersymmetry transformation of fermion and also supersymmetry parameters are no longer constant and vary on the 2-sphere. These two differences have two results. The first one appears in the closure of bosonic and fermionic fields that we have a SO(8) R-symmetry rotation. This rotation didn't appear for gauge fields because they have singlet representation of SO(8). Appearing a new term in the equation of motion for X's leaded a mass term in the Lagrangian is the second one. However the equations of motion for gauge and fermionic fields formally remain unchange. Our theory like original BLG theory is parity invariance as expected. We have also considered small fluctuation about vacuum and superalgebra .

It was argued that ABJM model can be written on $R\times S^2$. Although f^{abcd} is not real and fully antisymmetric the supersymmetry transformation including mass term closes up to a gauge transformation.

In the last section we have studied BPS configurations. One family of 1/4 BPS configurations are fuzzy three sphere with SO(4) symmetry and another one can be considered as M5-M2 configuration which M5 has been wrapped in the ϕ direction. In the Basu-Harvey limit this family of solutions reproduce Basu-Harvey configuration as an open membrane ending on M5-brane.

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A Gamma matrices

In this appendix we briefly consider our notation of Γ-matrices. The eleven dimensional Γ-matrices are defined by

$$\{\Gamma^M,\Gamma^N\}=2\eta^{MN},\quad M,N=0,\ldots,10 \tag{A.1}$$

where $\eta^{MN}=diag(-,+^{10})$. Under dimension reduction to three dimensions we have

$$SO(10,1)\supset SO(2,1)\times SO(8) \tag{A.2}$$

$$\begin{aligned}\{\gamma^\mu,\gamma^\nu\}&=2\eta^{\mu\nu},\quad \mu,\nu=0,1,2\\ \{\Gamma^I,\Gamma^J\}&=2\delta^{IJ},\quad I,J=1,\ldots,8\\ \{\Gamma^I,\gamma^\mu\}&=0\\ \gamma^\mu&=\bar{\gamma}^\mu\otimes\gamma^8\\ \Gamma^I&=\mathbb{1}_4\otimes\gamma^I\end{aligned} \tag{A.3}$$

where

$$\bar{\gamma}^\mu = \begin{pmatrix} 0 & i\tau^\mu \\ -i\tau^\mu & 0 \end{pmatrix}, \quad \tau^0 = i\sigma^3, \tau^1 = \sigma^1, \tau^2 = \sigma^2 \tag{A.4}$$
$$\{\gamma^I, \gamma^8\} = 0, \quad (\gamma^8)^2 = 1$$

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